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ON THE COMPUTATION OF STRESS INTENSITIES AT FIXED-FREE CORNERS

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Abstract—A path independent integral formula is developed for the computation of the intensity of the stress singularity at a right corner where one edge is rigidly fixed and the other is free of traction. Numerical results are presented for the case of a strip compressed between rough rigid stamps and compared with previously published results for finite and semi-infinite strips and cylinders.

INTRODUCTION

A problem of continuing interest is the calculation of the stress field in a cylinder or strip with prescribed tractions or displacements on its ends. In particular, when an end is rigidly fixed and the lateral surface is unrestrained, the stress field in the neighborhood of the corner becomes singular, and the strength and intensity of this singularity is of interest (see, for example [1–5]). In this note we utilize a reciprocal work contour integral method described earlier [6]† to develop a simple but efficient computational technique for evaluating the strength and intensity of such corner singularities in plane problems. It turns out that the strength of the singularity in the axisymmetric problem is independent of the cylinder radius and hence is the same as for the plane strain case we will treat here (see [2]). As will be seen later the stress intensities computed for similar problems in both cases are also generally comparable.

DERIVATION OF THE CONTOUR INTEGRAL FORMULA

We outline briefly the development of the singular elastic states needed in the reciprocal work representation. An origin of polar coordinates is placed at the corner with $\theta = 0$ the fixed edge and $\theta = \pi/2$ the traction free edge as indicated in Fig. 1. In terms of the complex variable $z = r\,e^{i\theta}$ the equations of plane isotropic elasticity for equilibrium configurations in the absence of body forces have a representation in terms of complex potentials $\Omega(z)$ and $\omega(z)$ ([8], Chap. VIII) in the form

$$\begin{aligned} U &= u_r + iu_\theta = (2\mu)^{-1} e^{-i\theta}[\kappa\Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}(\bar{z})] \\ T_r &= \sigma_{rr} + i\sigma_{r\theta} = \Omega'(z) + \bar{\Omega}'(\bar{z}) - \bar{z}\bar{\Omega}''(\bar{z}) - \bar{z}z^{-1}\bar{\omega}'(\bar{z}) \\ T_\theta &= \sigma_{\theta\theta} - i\sigma_{r\theta} = \Omega'(z) + \bar{\Omega}'(\bar{z}) + \bar{z}\bar{\Omega}''(\bar{z}) + \bar{z}z^{-1}\bar{\omega}'(\bar{z}) \end{aligned} \tag{1}$$

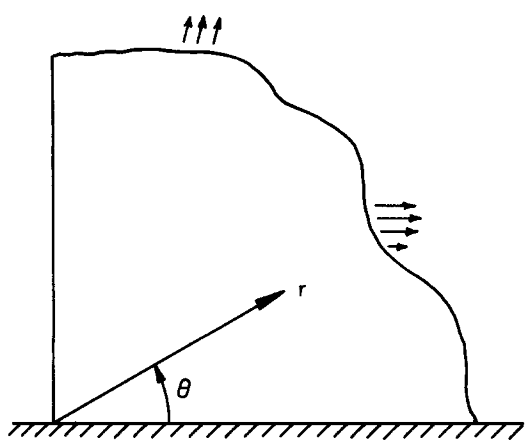

Fig. 1. Local coordinates for fixed-free corner.

†This method is also colsely related to the integral equation method described in [7].

where u_r, u_θ are physical displacement components, σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ are stress components, μ is the shear modulus, and $\kappa = 3 - 4\nu$ (for plane strain) with ν Poisson's ratio. Following the ideas of England[9], we seek nontrivial potentials of the form

$$\Omega(z) = Az^{\lambda}, \qquad \omega(z) = Bz^{\lambda} \tag{2}$$

which lead to fields satisfying the homogeneous conditions

$$U|_{\theta=0} = 0, \qquad T_\theta|_{\theta=\pi/2} = 0. \tag{3}$$

Upon substituting (2) into (1), the boundary conditions (3) require

$$\begin{aligned} \kappa A - \lambda \bar{A} - \bar{B} &= 0 \\ A\,\mathrm{e}^{i\lambda\pi} - \lambda\bar{A} + \bar{B} &= 0 \end{aligned} \tag{4}$$

which yields the characteristic equation

$$\cos \lambda\pi = \frac{2\lambda^2}{\kappa} - \frac{\kappa^2 + 1}{2\kappa}. \tag{5}$$

For λ an eigenvalue we have the associated eigenstate defined by

$$B = \kappa\bar{A} - \lambda A \tag{6}$$

where the real and imaginary parts of $A = a + ia'$ are related through

$$\alpha = \frac{a'}{a} = \frac{\kappa - 2\lambda + \cos\lambda\pi}{\sin\lambda\pi} = -\frac{\sin\lambda\pi}{\kappa + 2\lambda + \cos\lambda\pi} \tag{7}$$

provided λ is not an integer.

Denote by **u** the displacement field and **t** the traction vector on a contour C corresponding to the solution of a particular plane equilibrium problem in a simply connected region containing C, and let $\hat{\mathbf{u}}$, $\hat{\mathbf{t}}$ correspond to any other such solution. In the absence of body forces Betti's reciprocal work relation[10] takes the form

$$\int_C (\mathbf{u}\,.\,\hat{\mathbf{t}} - \hat{\mathbf{u}}\,.\,\mathbf{t})\,\mathrm{d}s = 0. \tag{8}$$

Referring to Fig. 2, since the stress field associated with any particular problem is singular at the corner we delete a small neighborhood of the singular point with the quarter arc C_ϵ and

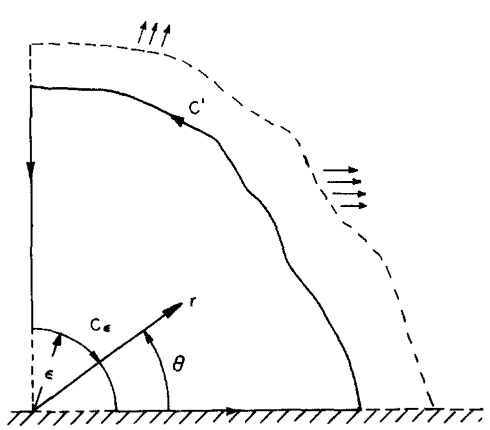

Fig. 2. Integration contour.

denote the remainder of the contour C by C'. Then eqn (8) becomes

$$-\int_{C_\epsilon} (\mathbf{u}\,.\,\hat{\mathbf{t}} - \hat{\mathbf{u}}\,.\,\mathbf{t})\,\mathrm{d}s = \int_{C'} (\mathbf{u}\,.\,\hat{\mathbf{t}} - \hat{\mathbf{u}}\,.\,\mathbf{t})\,\mathrm{d}s. \tag{9}$$

The idea now is to employ a suitable auxiliary elastic state so that the integral on C_ϵ can be evaluated for arbitrarily small ϵ in terms of the singular stress field intensity. In the neighborhood of the corner the displacement and stress fields are of the form†

$$\begin{aligned}
u_r &= \frac{r^\lambda}{\lambda(1+\kappa)}\{(\kappa-\lambda)K_I[\cos(1-\lambda)\theta - \cos(1+\lambda)\theta] \\
&\quad - K_{II}[(\kappa-\lambda)\sin(1-\lambda)\theta - (\kappa+\lambda)\sin(1+\lambda)\theta]\} + \text{remainder} \\
u_\theta &= \frac{r^\lambda}{\lambda(1+\kappa)}\{K_I[-(\kappa+\lambda)\sin(1-\lambda)\theta + (\kappa-\lambda)\sin(1+\lambda)\theta] \\
&\quad - (\kappa+\lambda)K_{II}[\cos(1-\lambda)\theta - \cos(1+\lambda)\theta]\} + \text{remainder} \\
\sigma_r &= \frac{r^{-(1-\lambda)}}{1+\kappa}\{K_I[(3-\lambda)\cos(1-\lambda)\theta - (\kappa-\lambda)\cos(1+\lambda)\theta] \\
&\quad - K_{II}[(3-\lambda)\sin(1-\lambda)\theta - (\kappa+\lambda)\sin(1+\lambda)\theta\} + \text{remainder} \\
\sigma_\theta &= \frac{r^{-(1-\lambda)}}{1+\kappa}\{K_I[(1+\lambda)\cos(1-\lambda)\theta + (\kappa-\lambda)\cos(1+\lambda)\theta] \\
&\quad - K_{II}[(1+\lambda)\sin(1-\lambda)\theta + (\kappa+\lambda)\sin(1+\lambda)\theta]\} + \text{remainder} \\
\tau_{r\theta} &= \frac{r^{-(1-\lambda)}}{1+\kappa}\{K_I[(1-\lambda)\sin(1-\lambda)\theta + (\kappa-\lambda)\sin(1+\lambda)\theta] \\
&\quad + K_{II}[(1-\lambda)\cos(1-\lambda)\theta + (\kappa+\lambda)\cos(1+\lambda)\theta]\} + \text{remainder}
\end{aligned} \tag{10}$$

where the remainder terms are higher order than the dominant singular terms shown. The value of λ is the smallest non-negative root of the characteristic eqn (5) and the principal part of the elastic state (10) is then precisely the eigenstate corresponding to this value of λ. (The least positive root of eqn (5) lies between $\lambda = 0{\cdot}5946$ for an incompressible material in plane strain, and unity as Poisson's ratio decreases to zero.) Furthermore, we have introduced the conventional stress intensity factors

$$\begin{aligned}
K_I &= \lim_{r\to 0} r^{-(1-\lambda)}\sigma_\theta|_{\theta=0} \\
K_{II} &= \lim_{r\to 0} r^{-(1-\lambda)}\sigma_{r\theta}|_{\theta=0}
\end{aligned} \tag{11}$$

and to be consistent with eqns (6) and (7) we have

$$K_{II}/K_I = -\alpha = \frac{\sin\lambda\pi}{\kappa + 2\lambda + \cos\lambda\pi} \tag{12}$$

for all loadings.

Since $-\lambda$ must also be an eigenvalue of (5), the auxiliary state is taken to be the corresponding eigenstate

$$\begin{aligned}
\hat{u}_r &= \frac{cr^{-\lambda}}{\lambda(1+\kappa)}\{(\kappa+\lambda)[\cos(1-\lambda)\theta - \cos(1+\lambda)\theta] \\
&\quad + \gamma[(\kappa-\lambda)\sin(1-\lambda)\theta - (\kappa+\lambda)\sin(1+\lambda)\theta]\}
\end{aligned}$$

†See [6] or [9].

$$\hat{u}_\theta = \frac{cr^{-\lambda}}{\lambda(1+\kappa)}\{-[(\kappa+\lambda)\sin(1-\lambda)\theta-(\kappa-\lambda)\sin(1+\lambda)\theta]$$
$$+\gamma(\kappa-\lambda)[\cos(1-\lambda)\theta-\cos(1+\lambda)\theta]\}$$
$$\hat{\sigma}_r = \frac{cr^{-(1+\lambda)}}{1+\kappa}\{[(\kappa+\lambda)\cos(1-\lambda)\theta-(3+\lambda)\cos(1+\lambda)\theta]$$
$$+\gamma[(\kappa-\lambda)\sin(1-\lambda)\theta-(3+\lambda)\sin(1+\lambda)\theta]\}$$
$$\hat{\sigma}_\theta = \frac{cr^{-(1+\lambda)}}{1+\kappa}\{[(\kappa+\lambda)\cos(1-\lambda)\theta+(1-\lambda)\cos(1+\lambda)\theta]$$
$$+\gamma[(\kappa-\lambda)\sin(1-\lambda)\theta+(1-\lambda)\sin(1+\lambda)\theta]\}$$
$$\hat{\sigma}_{r\theta} = \frac{cr^{-(1+\lambda)}}{1+\kappa}\{[(\kappa+\lambda)\sin(1-\lambda)\theta+(1+\lambda)\sin(1+\lambda)\theta]$$
$$-\gamma[(\kappa-\lambda)\cos(1-\lambda)\theta+(1+\lambda)\cos(1+\lambda)\theta]\} \quad (13)$$

where $\gamma = 1/\alpha$, c is an arbitrary constant, and we note that $\hat{U}$ vanishes on $\theta = 0$ while $\hat{T}_\theta$ vanishes on $\theta = \pi/2$.

The evaluation of the integral on the left in eqn (9) for arbitrarily small ϵ is direct although somewhat tedious

$$I = -\lim_{\epsilon\to 0}\int_{C_\epsilon}(\mathbf{u}\,.\,\hat{\mathbf{t}}-\hat{\mathbf{u}}\,.\,\mathbf{t})\,\mathrm{d}s$$
$$= \lim_{\epsilon\to 0}\int_0^{\pi/2}[(u_r\hat{\sigma}_r+u_\theta\hat{\sigma}_{r\theta})-(\hat{u}_r\sigma_r+\hat{u}_\theta\sigma_{r\theta})]_{r=\epsilon}\,\epsilon\,\mathrm{d}\theta$$
$$= \frac{2c\beta}{\lambda(1+\kappa)^2}K_I \quad (14)$$

where

$$\beta = \left[\frac{\kappa(\kappa+1)}{\lambda}-\frac{\lambda(4+3\lambda)}{1+\lambda}\right]\sin\lambda\pi + 2\lambda(3\kappa-\cos\lambda\pi)\cot\lambda\pi/2$$
$$+\frac{\kappa^2+2\lambda^2-\kappa}{\lambda}(\kappa+\cos\lambda\pi)\tan\lambda\pi/2-\kappa(\kappa+1)\pi. \quad (15)$$

The remaining integral is considerably simplified by having chosen the auxiliary state to satisfy the homogeneous boundary conditions on $\theta = 0$ and $\theta = \pi/2$ since the integrands vanish on these lines.† Thus we are left with the representation

$$K_I = \frac{\lambda(1+\kappa)^2}{2\beta}\int_{C'}[\mathbf{u}\,.\,(\hat{\mathbf{t}}/c)-(\hat{\mathbf{u}}/c)\,.\,\mathbf{t}]\,\mathrm{d}s \quad (16)$$

where C' is any convenient contour connecting the edges $\theta = 0$ and $\theta = \pi/2$ in the body and $\hat{\mathbf{u}}/c$, $\hat{\mathbf{t}}/c$ are directly computable from eqn (13). We are still left with the necessity of evaluating the traction and displacement on C', but because this contour may be chosen remote from the singularity, we need not be concerned with an accurate representation of stresses in the neighborhood of the corner. This permits the use of any standard finite element code to compute tractions and displacements on C' without requiring an expensive mesh refinement at the corner.

COMPUTATIONAL RESULTS

The evaluation of the contour integral was easily incorporated into the finite element code TEXGAP[11] which uses a conventional displacement method to perform two dimensional

†Should the edge $\theta = \pi/2$ be loaded with a prescribed traction $\bar{\mathbf{t}}$ we merely add the contribution $\int_{\theta=\pi/2} -\hat{\mathbf{u}}\,.\,\bar{\mathbf{t}}\,\mathrm{d}s$ which may be evaluated in any convenient manner.

linearly elastic analyses. Details are entirely similar to those described in [12] where the analogous computation for crack tip stress intensities was treated.

To illustrate the nature of the computation and the quality of the results obtainable we treat the problem of a strip compressed between rough rigid stamps as indicated in Fig. 3 and compare results with those published by Benthem[3] and Gupta[4, 5] for the strip, and by Benthem and Minderhoud[1] and Gupta[12] for the analogous axisymmetric cylinder problem.

From symmetry consideration only one quarter of the strip was subjected to stress analysis and the 30 element grid used is also shown in Fig. 3. The contour used in evaluating the integral of eqn (16) is indicated by the heavy dashed lines. In addition to the stress analysis performed by the finite element code TEXGAP, the characteristic eqn (5) had to be solved for the appropriate value of λ, the auxiliary state displacements and tractions computed from eqn (13) at points on the contour, and the results of a (5-point Gauss-Legendre) numerical quadrature of eqn (16) evaluated and accumulated for each element side in the contour. It should be noted however, that the additional central processor time needed for this computation is negligible compared to the running time for the finite element computations.

The strength of the singularity, λ, and the ratio K_{II}/K_I depend only on Poisson's ratio and may be computed directly from eqns (5) and (12). Since this is discussed adequately in [4] and [5] we will only comment that the dependence of λ and K_{II}/K_I on Poisson's ratio is identical for the axisymmetric cylinder and the plane strain strip.

In Fig. 4 we have plotted the dependence of the singularity intensity on Poisson's ratio for a relatively long strip ($L/h = 8{\cdot}0$); these results are indistinguishable from those given graphically by Gupta[4]† for a semi-infinite strip using an integral equation approach. Shown in the same figure are results obtained by Gupta[2] for the corresponding axisymmetric cylinder problem. The difference at $\nu = 0{\cdot}3$ is only about 8%.

The effect of the length to width ratio of the strip on the stress intensity at the corner is indicated in Fig. 5. There does appear to be a significant difference between the present results

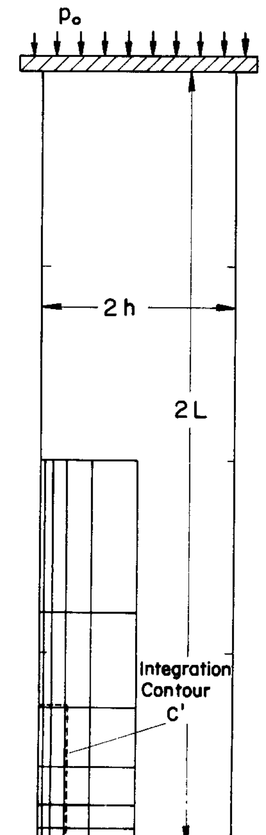

Fig. 3. Grid for compression strip.

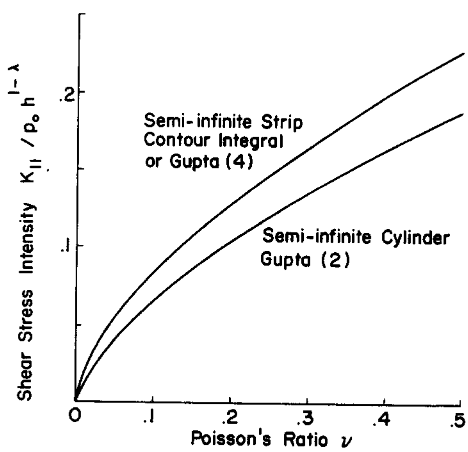

Fig. 4. Dependence of stress intensity on Poisson's ratio.

†The reader is cautioned that the definition of K_I and K_{II} given in [2, 4, 5] contains a factor of $\sqrt{2}$ not present here.

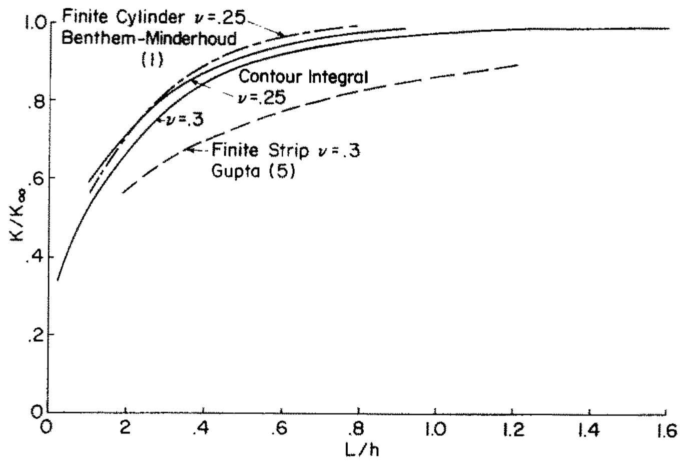

Fig. 5. Stress intensity for finite strips.

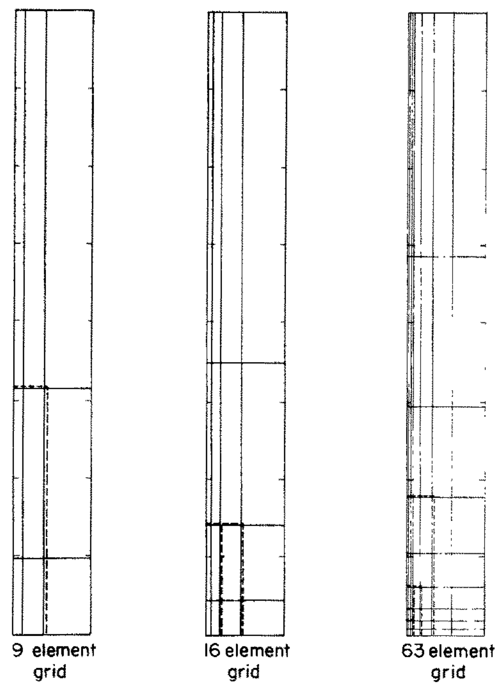

Fig. 6. Finite element grids.

Table 1. Values of $K_{II}/p_0h^{1-\lambda}$ for various grids and contours $L/h = 8$, $\nu = 0{\cdot}3$

	63 elem. Grid	30 elem. Grid	16 elem. Grid	9 elem. Grid
4 elem. contour	0·165	0·168	0·168	0·169
6 elem. contour	0·163	0·164	0·164	
12 elem. contour	0·163	0·163		
16 elem. contour	0·163			
30 elem. contour	0·163			

and those given by Gupta[5], but the cause of this discrepency is not clear. Also shown in Fig. 5 are results published by Benthem and Minderhoud[1] for the analogous effect of the length to diameter ratio for an axisymmetric cylinder. This curve would tend to substantiate the values obtained by the present method.

Finally, as an indication of the convergence and accuracy of the method, and its relative independence on contour selection we treated the 9, 16 and 63 element grids shown in Fig. 6. Furthermore, results were obtained using various contours on each grid, and these findings are tabulated in Table 1.

As can be seen, the method is reasonably accurate and efficient, and not very sensitive to choice of contour so long as it is a few elements removed from the singularity.

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REFERENCES

1. J. P. Benthem and P. Minderhoud, The problem of the solid cylinder compressed between rigid stamps. *Int. J. Solids Structures* **8**, 1027 (1972).
2. G. D. Gupta, The analysis of the semi-infinite cylinder problem. *Int. J. Solids Structures* **10**, 137 (1974).
3. J. P. Benthem, A Laplace transform method for the solution of semi-infinite and finite strip problems in stress analysis. *Quarterly J. Mech. and Applied Math.* **16**, 413 (1963).
4. G. D. Gupta, An integral equation approach to the semi-infinite strip problem. *J. Applied Mech.* **40**, 948 (1973).
5. G. D. Gupta, The problem of a finite strip compressed between two rigid stamps. *ASME* paper No. 75-APMW-13.
6. M. Stern, A boundary integral representation for stress intensity factors. *Proc. 10th Anniv. Meeting, Soc. of Eng. Sci.*, Raleigh, N.C., 5–7, Nov. 1973.
7. M. R. Barone and A. R. Robinson, Determination of elastic stresses at notches and corners by integral equations. *Int. J. Solids Structures* **8**, 1319 (1972).
8. A. E. Green and W. Zerna, *Theoretical Elasticity*, 2nd Edn. Oxford University Press (1968).
9. A. H. England, On stress singularities in linear elasticity. *Int. J. Engr. Science* **9**, 571 (1971).
10. I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd Edn. McGraw-Hill, New York (1965).
11. R. S. Dunham and E. B. Becker, TEXGAP—the Texas grain analysis program. *TICOM Report* 73–1. The University of Texas at Austin, August 1973.
12. M. Stern, E. B. Becker and R. S. Dunham, A contour integral computation of mixed mode stress intensity factors. *Int. J. Fracture.* To be published.